

ABSOLUTE CONTINUITY AND α -NUMBERS ON THE REAL LINE

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ABSTRACT. Let μ, ν be Radon measures on \mathbb{R} . For an interval $I \subset \mathbb{R}$, define $\alpha_{\mu, \nu}(I) := \mathbb{W}_1(\mu_I, \nu_I)$, the Wasserstein distance of normalised blow-ups of μ and ν restricted to I . Let S_ν be either one of the square functions

$$S_\nu^2(\mu)(x) = \sum_{x \in I \in \mathcal{D}} \alpha_{\mu, \nu}^2(I) \quad \text{or} \quad S_\nu^2(\mu)(x) = \int_0^1 \alpha_{\mu, \nu}^2(B(x, r)) \frac{dr}{r},$$

where \mathcal{D} is the family of dyadic intervals of side-length at most one. Assuming that ν is doubling, and μ does not charge the boundaries of intervals in \mathcal{D} (in the case of the first square function), I prove that $\mu|_G \ll \nu$, where

$$G = \{x : S_\nu(\mu)(x) < \infty\}.$$

If μ is also doubling, and the square functions satisfy suitable Carleson conditions, then absolute continuity can be improved to $\mu \in A_\infty(\nu)$. The results answer the simplest “ $n = d = 1$ ” case of a problem of J. Azzam, G. David and T. Toro.

1. INTRODUCTION

1.1. Main results. In this paper, μ and ν are non-zero Radon measures. The measure ν is generally assumed to be either *dyadically doubling* or *globally doubling*. Dyadically doubling means that

$$\nu(\hat{I}) \leq C\nu(I), \quad I \in \mathcal{D}, \tag{1.1}$$

where \mathcal{D} is the standard family of dyadic intervals, and \hat{I} is the *parent* of I , that is, the smallest interval in \mathcal{D} strictly containing I . Globally doubling means that $\nu(B(x, 2r)) \leq C\nu(B(x, r))$ for $x \in \mathbb{R}$ and $r > 0$; in particular, this implies $\text{spt } \nu = \mathbb{R}$. The main example for ν is the Lebesgue measure \mathcal{L} , and the proofs in this particular case would differ little from the ones presented below. No *a priori* homogeneity is assumed of μ .

Definition 1.2 (Wasserstein distance). I will use the following definition of the (first) Wasserstein distance: given two Radon measures ν_1, ν_2 on $[0, 1]$, set

$$\mathbb{W}_1(\nu_1, \nu_2) := \sup_{\psi} \left| \int \psi d\nu_1 - \int \psi d\nu_2 \right|,$$

where the sup is taken over all 1-Lipschitz functions $\psi: \mathbb{R} \rightarrow \mathbb{R}$, which are supported on $[0, 1]$. Such functions will be called *test functions*. A slightly different – and equally common – definition would allow the sup to run over all 1-Lipschitz functions $\psi: [0, 1] \rightarrow \mathbb{R}$. To illustrate the difference, let $\nu_1 = \delta_0$ and $\nu_2 = \delta_1$. Then $\mathbb{W}_1(\nu_1, \nu_2) = 0$, but the alternative definition, say $\tilde{\mathbb{W}}_1$, would give $\tilde{\mathbb{W}}_1(\nu_1, \nu_2) = 1$. The only reason for using the definition \mathbb{W}_1 in this paper is to make the results as strong as possible.

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As in the paper [1] of J. Azzam, G. David and T. Toro, I make the following definition:

Definition 1.3 (α -numbers). Assume that $I \subset \mathbb{R}$ is an interval. Define

$$\alpha_{\mu,\nu}(I) := \mathbb{W}_1(\mu_I, \nu_I),$$

where μ_I and ν_I are normalised blow-ups of μ and ν restricted to I . More precisely, let T_I be the increasing affine mapping taking \bar{I} to $[0, 1]$, and define

$$\mu_I := \frac{T_{I\#}(\mu|_I)}{\mu(I)} \quad \text{and} \quad \nu_I := \frac{T_{I\#}(\nu|_I)}{\nu(I)}.$$

If $\mu(I) = 0$ (or $\nu(I) = 0$), define $\mu_I \equiv 0$ (or $\nu_I \equiv 0$).

Remark 1.4. A variant of the α -numbers was first introduced by X. Tolsa in [7], where he used them to characterise the uniform rectifiability of Ahlfors-David regular measures in \mathbb{R}^n . Tolsa's original definition was crafted only to compute distances between Ahlfors-David regular measures, so it has a different "normalisation" than the one above.

Before explaining the results in Azzam, David and Toro's paper [1], and their connection to the current manuscript, I want to emphasise that [1] deals with " d -dimensional" measures in \mathbb{R}^n , for any $1 \leq d \leq n$. For the current paper, only the case $n = d = 1$ is relevant. So, to avoid digressing too much, I need to state the results of [1] in far smaller generality than they would actually deserve.

With this proviso in mind, the main results of [1] imply the following. if μ is a doubling measure on \mathbb{R} , and the numbers $\alpha_{\mu,\mathcal{L}}$ satisfy a Carleson condition of the form

$$\int_{B(x,2r)} \int_0^{2r} \alpha_{\mu,\mathcal{L}}(B(y,t)) \frac{dt d\mu y}{t} \leq C\mu(B(x,r)), \quad (1.5)$$

then μ , or at least a large part of μ , is absolutely continuous with respect to \mathcal{L} , with quantitative upper and lower bounds on the density. As the authors of [1] point out, the main shortcoming of their result is that condition (1.5) imposes a hypothesis on the first powers of the numbers $\alpha_{\mu,\mathcal{L}}$, whereas evidence suggests (see [1, Remark 1.6.1], the discussion after [1, Theorem 1.7], and [1, Example 4.6]) that the correct power should be two. More support for this conjecture comes from the following result of Tolsa [8, Theorem 1.3]: if $\mu \in L^1(\mathbb{R})$, then

$$\int_0^\infty \tilde{\alpha}_{\mu,\mathcal{L}}^2(x,r) \frac{dr}{r} < \infty \quad \text{at } \mu \text{ a.e. } x \in \mathbb{R}.$$

I should again mention that this is only the easiest $n = d = 1$ case of Tolsa's theorem. Also, $\tilde{\alpha}_{\mu,\mathcal{L}}$ is a slightly different variant of the α -number, but it seems very likely that the result is true with the present paper's conventions.

The purpose of this paper is to address the problem of Azzam, David and Toro in the simplest case $n = d = 1$. I show that control for the second powers of the $\alpha_{\mu,\mathcal{L}}$ -numbers does, indeed, guarantee absolute continuity with respect to Lebesgue measure. In fact, the doubling assumption on μ can be dropped, the Carleson condition (1.5) can be relaxed considerably, and the results remain valid, if \mathcal{L} is replaced by any doubling measure ν .

I prove two variants of the main result: a dyadic, and a continuous, one. Here is the dyadic version:

Theorem 1.6. Assume that μ, ν are Borel probability measures on $[0, 1)$. Assume that μ does not charge the boundaries of dyadic intervals, and ν is dyadically doubling on subintervals of $[0, 1)$. Then, $\mu|_G \ll \nu$, where

$$G := \{x : \mathcal{S}_{\mathcal{D}, \nu}(\mu)(x) < \infty\},$$

and $\mathcal{S}_{\mathcal{D}, \nu}$ is the square function

$$\mathcal{S}_{\mathcal{D}, \nu}^2(\mu) = \sum_{I \in \mathcal{D}} \alpha_{\mu, \nu}^2(I) \chi_I.$$

In particular,

$$\sum_{I \in \mathcal{D}} \alpha_{\mu, \nu}^2(I) \mu(I) < \infty \implies \mu \ll \nu.$$

Heuristically, this corresponds to assuming (1.5) only at the scale $r = 1$, but I have not found a way to **reduce** the continuous problem to the dyadic one; on the other hand, a reduction in the other direction does not appear easy, either, so perhaps one just needs to treat the cases separately. A caveat of the dyadic set-up is the "non-atomicity" hypothesis on μ . It cannot be dispensed with: for instance, if $\mu = \delta_x$ for any $x \in [0, 1)$, which only belongs to the interiors of finitely many dyadic intervals, then $\mathcal{S}_{\mathcal{D}, \mathcal{L}}(\mu)$ is uniformly bounded (for instance $\mathcal{S}_{\mathcal{D}, \mathcal{L}}(\delta_0) \equiv 0$), but clearly $\mu \not\ll \mathcal{L}$. The simplest solution would be to consider the Wasserstein distance $\tilde{\mathbb{W}}_1$, but I will not pursue this matter further.

Here is the continuous version of the main theorem:

Theorem 1.7. Assume that μ, ν are Radon measures, and ν is globally doubling. Then, $\mu|_G \ll \nu$, where

$$G := \{x : \mathcal{S}_\nu(\mu)(x) < \infty\},$$

and \mathcal{S}_ν is the square function

$$\mathcal{S}_\nu^2(\mu)(x) = \int_0^1 \alpha_{\mu, \nu}^2(B(x, r)) \frac{dr}{r}, \quad x \in \text{spt } \mu.$$

In case $\text{spt } \mu \neq \mathbb{R}$, the assumption on ν being globally doubling could be easily relaxed to being doubling in a neighbourhood of $\text{spt } \mu$. In particular, if the Carleson condition (1.5) holds for some point x_0 and radius $r_0 > 0$ such that $\text{spt } \mu \subset B(x_0, r_0/2)$, and ν is doubling in $B(x_0, r_0)$, then $\mu \ll \nu$.

Finally, assuming the full Carleson condition (1.5), and that μ is globally doubling, the authors of [1] prove something more quantitative than $\mu \ll \mathcal{L}$; see in particular [1, Theorem 1.9]. The same ought to be true for the second powers of the α -numbers, and indeed the following result can be easily deduced with the method of the current paper:

Theorem 1.8. Assume that μ, ν are Borel probability measures on $[0, 1)$, both dyadically doubling on sub-intervals of $[0, 1)$, and assume that the Carleson condition

$$\sum_{I \subset J} \alpha_{\mu, \nu}^2(I) \mu(I) \leq C \mu(J), \quad J \in \mathcal{D}, \quad (1.9)$$

holds for some $C \geq 1$. Then μ lies in the class $A_\infty^\mathcal{D}(\nu)$, the dyadic A_∞ class relative to ν . Similarly, if μ, ν are Radon measures on \mathbb{R} , both globally doubling, and the Carleson condition (1.5) holds for the second powers $\alpha_{\mu, \nu}^2(B(y, t))$, then $\mu \in A_\infty(\nu)$.

The *a priori* doubling assumptions cannot be omitted (that is, they are not implied by the Carleson condition): just consider $\mu = 2\chi_{[0,1/2)} d\mathcal{L}$. It is clear that the Carleson condition (1.9) holds for the numbers $\alpha_{\mu,\mathcal{L}}^2(I)$, but nevertheless $\mu \notin A_\infty^{\mathcal{D}}(\mathcal{L}|_{[0,1]})$.

1.2. Outline of the proofs, and the paper. Most of the paper is concerned with the proof of the dyadic result, Theorem 1.6; it is both conceptually and technically a little simpler than the proof of the "continuous" variant, Theorem 1.7, given in Section 5. There is just one not-purely-technical difference between the arguments: the continuous case seems to require (or at least benefit greatly from) working initially with "smooth" variants of the α -numbers, to be introduced at the start of Section 5. These are (often) dominated by the "real" α -numbers, but enjoy increased stability.

The proof of Theorem 1.6 has three main steps. First, the unwieldy numbers $\alpha_{\mu,\nu}(I)$ are used to control something analyst-friendlier, namely the following dyadic variants:

$$\Delta_{\mu,\nu}(I) = \left| \frac{\mu(I_-)}{\mu(I)} - \frac{\nu(I_-)}{\nu(I)} \right|. \quad (1.10)$$

Here I_- stands for the left half of I . This would be simple, if $\chi_{[0,1/2)}$ happened to be one of the admissible test functions ψ in the definition of \mathbb{W}_1 . It is not, however, and in fact there seems to be no direct (and sufficiently efficient) way to control $\Delta_{\mu,\nu}(I)$ by $\alpha_{\mu,\nu}(I)$, or even $\alpha_{\mu,\nu}(3I)$. However, it turns out that the numbers are equivalent at the level of certain Carleson sums over trees; proving this statement is the main content of Section 2.

The numbers $\Delta_{\mu,\nu}(I)$ are well-known quantities: they are the (absolute values of the) coefficients in an orthogonal representation of μ in terms of ν -adapted Haar functions, and it is known that they can be used to characterise A_∞ . The following theorem is due to S. Buckley [3] from 1993:

Theorem 1.11 (Theorem 2.2(iii) in [3]). *Let μ, ν be a dyadically doubling Borel probability measures on $[0, 1]$. Then $\mu \in A_\infty^{\mathcal{D}}(\nu)$, if and only if*

$$\sum_{I \subset J} \Delta_{\mu,\nu}^2(I) \mu(I) \leq C \mu(J), \quad J \in \mathcal{D}. \quad (1.12)$$

The result in [3] is only stated for $\nu = \mathcal{L}|_{[0,1]}$, but the proof works in the greater generality. Note the similarity between the Carleson conditions (1.12) and (1.9): The dyadic part of Theorem 1.8 is, in fact, nothing but a corollary of Buckley's result, assuming that one knows how to control the numbers $\Delta_{\mu,\nu}(I)$ by the numbers $\alpha_{\mu,\nu}(I)$ at the level of Carleson sums; consequently, the short proof of this half of Theorem 1.8 can be found in Section 2. The "continuous" part is discussed at the end of the paper, in Remark 5.20.

Buckley's result is not applicable for Theorem 1.6: the measure μ is not dyadically doubling, and the information available is much weaker than the Carleson condition (1.9). Handling these issues constitutes the remaining two steps in the proof: all dyadic intervals are split into trees, where μ is "tree-doubling" (Section 4), and the absolute continuity of μ with respect to ν is studied in each tree separately (Section 3).

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2. COMPARISON OF α -NUMBERS AND Δ -NUMBERS

In this section, μ and ν are Borel probability measures on $[0, 1]$, μ does not charge the boundaries of dyadic intervals, and ν is dyadically doubling inside $[0, 1]$:

$$\nu(\hat{I}) \leq D_\nu \nu(I), \quad I \in \mathcal{D} \setminus \{[0, 1]\}.$$

This implies, in particular, that $\nu(I) > 0$ for all $I \in \mathcal{D}$ with $I \subset [0, 1]$. The main task of the section is to bound the numbers $\Delta_{\mu, \nu}(I)$ by the numbers $\alpha_{\mu, \nu}(I)$, where $\Delta_{\mu, \nu}(I)$ was the quantity

$$\Delta_{\mu, \nu}(I) = \left| \frac{\mu(I_-)}{\mu(I)} - \frac{\nu(I_-)}{\nu(I)} \right| = \left| \int \chi_{(0, 1/2)} d\mu_I - \int \chi_{(0, 1/2)} d\nu_I \right|.$$

The task would be trivial, if $\chi_{(0, 1/2)}$ were a 1-Lipschitz function vanishing at the boundary of $[0, 1]$. It is not: in fact, the difference between $\Delta_{\nu_1, \nu_2}(I)$ and $\alpha_{\nu_1, \nu_2}(I)$ can be rather large for a given interval I .

Example 2.1. If $\nu_1 = \delta_{1/2-1/n}$ and $\nu_2 = \delta_{1/2+1/n}$, then $\Delta_{\nu_1, \nu_2}([0, 1]) = 1$, but $\alpha_{\nu_1, \nu_2}([0, 1]) \lesssim 1/n$. These measures do not satisfy the assumptions of the section, so consider also the following example. Let $\mu = f d\mathcal{L}$, where f takes the value 1 everywhere, except in the 2^{-n} -neighbourhood of $1/2$. Let $f \equiv 1/2$ on the interval $[1/2 - 2^{-n}, 1/2]$, and $f \equiv 3/2$ on the interval $(1/2, 1/2 + 2^{-n}]$. Then μ is dyadically 4-doubling probability measure on $[0, 1]$, $\Delta_{\mu, \mathcal{L}}([0, 1]) \sim 2^{-n}$, and $\alpha_{\mu, \mathcal{L}}([0, 1]) \sim 2^{-2n}$.

Fortunately, "pointwise" estimates between $\Delta_{\mu, \nu}(I)$ and $\alpha_{\mu, \nu}(I)$ are not really needed in this paper, and it turns out that certain sums of these numbers are comparable, up to a manageable error. To state such results, I need to introduce some terminology. A family $\mathcal{C} \subset \mathcal{D}$ of dyadic intervals is called *coherent*, if the implication

$$Q, R \in \mathcal{C} \text{ and } Q \subset P \subset R \implies P \in \mathcal{C}$$

holds for all $Q, P, R \in \mathcal{D}$.

Definition 2.2 (Trees, leaves, boundary). A *tree* $\mathcal{T} \subset \mathcal{D}$ is any coherent family of dyadic intervals with a unique largest interval, $\mathbf{Top}(\mathcal{T}) \in \mathcal{T}$, and with the property that

$$\text{card}(\mathbf{ch}(I) \cap \mathcal{T}) \in \{0, 2\}, \quad I \in \mathcal{T}.$$

For the tree \mathcal{T} , define the set family $\mathbf{Leaves}(\mathcal{T})$ to consist of the minimal intervals in \mathcal{T} , in other words those $I \in \mathcal{T}$ with $\text{card}(\mathbf{ch}(I) \cap \mathcal{T}) = 0$. Abusing notation, I often write $\mathbf{Leaves}(\mathcal{T})$ also for the set $\cup\{I : I \in \mathbf{Leaves}(\mathcal{T})\}$. Finally, define the *boundary* of the tree $\partial\mathcal{T}$ by

$$\partial\mathcal{T} := \mathbf{Top}(\mathcal{T}) \setminus \mathbf{Leaves}(\mathcal{T}).$$

Then $x \in \partial\mathcal{T}$, if and only if $x \in \mathbf{Top}(\mathcal{T})$, and all intervals $I \in \mathcal{D}$ with $x \in I \subset \mathbf{Top}(\mathcal{T})$ are contained \mathcal{T} .

Definition 2.3 ((\mathcal{T}, D) -doubling measures). A Borel probability measure μ on $[0, 1]$ is called (\mathcal{T}, D) -doubling, if

$$\mu(\hat{I}) \leq D\mu(I), \quad I \in \mathcal{T} \setminus \mathbf{Top}(\mathcal{T}).$$

Here is the main result of this section:

Proposition 2.4. *Let μ, ν be measures satisfying the assumptions of the section, and let $\mathcal{T} \subset \mathcal{D}$ be a tree. Moreover, assume that μ is (\mathcal{T}, D) -doubling for some constant $D \geq 1$. Then*

$$\sum_{I \in \mathcal{T}} \Delta_{\mu, \nu}^2(I) \mu(I) \lesssim_{D, D} \sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \alpha_{\mu, \nu}^2(I) \mu(I) + \mu(\text{Top}(\mathcal{T})).$$

The "dyadic part" of Theorem 1.8 is an immediate corollary:

Proof of Theorem 1.8, dyadic part. By hypothesis, both measures μ and ν are (\mathcal{D}, C) -doubling. Hence, by the Carleson condition (1.9), and Proposition 2.4 applied to the trees $\mathcal{T}_J := \{I \in \mathcal{D} : I \subset J\}$, one has

$$\sum_{I \subset J} \Delta_{\mu, \nu}^2(I) \mu(I) \lesssim_C \sum_{I \subset J} \alpha_{\mu, \nu}^2(I) \mu(I) + \mu(J) \lesssim \mu(J).$$

This is precisely the condition in Buckley's result, Theorem 1.11, so $\mu \in A_\infty^{\mathcal{D}}(\nu)$. \square

I then begin the proof of Proposition 2.4. It would, in fact, suffice to assume that ν is also just (\mathcal{T}, D_ν) -doubling, but checking this would result in some unnecessary book-keeping below. The proof is based on the observation that $\chi_{(0,1/2)}$ can be written as a series of Lipschitz functions, each supported on sub-intervals of $[0, 1]$. This motivates the following considerations.

Assume that

$$\Psi := \Psi_0 := \sum_{j \geq 0} \psi_j$$

is a bounded function such that each $\psi_j : \mathbb{R} \rightarrow [0, \infty)$ is an L_j -Lipschitz function supported on some interval $I_j \in \mathcal{D}_j$. Assume moreover that the intervals I_j are nested: $[0, 1] \supset I_1 \supset I_2 \dots$. Then, as a first step in proving Proposition 2.4, I claim that

$$\begin{aligned} \left| \int \Psi d\mu - \int \Psi d\nu \right| &\leq \sum_{k=0}^N \frac{L_k}{2^k} \alpha_{\mu, \nu}(I_k) \mu(I_k) \\ &\quad + \sum_{k=0}^N \left(\frac{1}{\nu(I_{k+1})} \int \Psi_{k+1} d\nu \right) \Delta_{\mu, \nu}(I_k) \mu(I_k) + 2\|\Psi\|_\infty \mu(I_{N+1}) \end{aligned} \quad (2.5)$$

for any $N \in \{0, 1, \dots, \infty\}$, where

$$\Psi_k := \sum_{j \geq k} \psi_j, \quad m \geq 0.$$

For $N = \infty$, the symbol " I_{N+1} " should be interpreted as the intersection of all the intervals I_j . I will first verify that, for any $m \geq 0$,

$$\begin{aligned} &\left| \frac{1}{\mu(I_m)} \int \Psi_m d\mu - \frac{1}{\nu(I_m)} \int \Psi_m d\nu \right| \\ &\leq \frac{L_m}{2^m} \alpha_{\mu, \nu}(I_m) + \left(\frac{1}{\nu(I_{m+1})} \int \Psi_{m+1} d\nu \right) \Delta_{\mu, \nu}(I_m) \\ &\quad + \frac{\mu(I_{m+1})}{\mu(I_m)} \left| \frac{1}{\mu(I_{m+1})} \int \Psi_{m+1} d\mu - \frac{1}{\nu(I_{m+1})} \int \Psi_{m+1} d\nu \right| \end{aligned} \quad (2.6)$$

from which it will be easy to derive (2.5). If $\mu(I_m) = 0$, the corresponding term should be interpreted as "0" (recall that $\nu(I_m)$ is never zero by the doubling assumption). The proof

of (2.6) is straightforward. First, note that since $\psi_m: \mathbb{R} \rightarrow \mathbb{R}$ is an L_m -Lipschitz function supported on I_m , and $|I_m| = 2^{-m}$, one has

$$\left| \frac{1}{\mu(I_m)} \int \psi_m d\mu - \frac{1}{\nu(I_m)} \int \psi_m d\nu \right| = \left| \int \psi_m \circ T_{I_m}^{-1} d\mu_{I_m} - \int \psi_m \circ T_{I_m}^{-1} d\nu_{I_m} \right| \leq \frac{L_m}{2^m} \alpha_{\mu, \nu}(I_m).$$

(The mappings T_I are familiar from Definition 1.3). This gives rise to the first term in (2.6). What remains is bounded by

$$\begin{aligned} & \left| \frac{1}{\mu(I_m)} \int \Psi_{m+1} d\mu - \frac{1}{\nu(I_m)} \int \Psi_{m+1} d\nu \right| \\ & \leq \frac{\mu(I_{m+1})}{\mu(I_m)} \left| \frac{1}{\mu(I_{m+1})} \int \Psi_{m+1} d\mu - \frac{1}{\nu(I_{m+1})} \int \Psi_{m+1} d\nu \right| \\ & \quad + \left(\frac{1}{\nu(I_{m+1})} \int \Psi_{m+1} d\nu \right) \left| \frac{\mu(I_{m+1})}{\mu(I_m)} - \frac{\nu(I_{m+1})}{\nu(I_m)} \right|. \end{aligned}$$

This is (2.6), observing that

$$\Delta_{\mu, \nu}(I_m) = \left| \frac{\mu(I_{m+1})}{\mu(I_m)} - \frac{\nu(I_{m+1})}{\nu(I_m)} \right|,$$

since either $I_{m+1} = (I_m)_+$ or $I_{m+1} = (I_m)_-$, and both possibilities give the same number $\Delta_{\mu, \nu}(I_m)$. Finally, (2.5) is obtained by repeated application of (2.6). By induction, one can check that N iterations of (2.6) (starting from $m = 0$, and recalling that μ, ν are probability measures on $[0, 1]$) leads to

$$\begin{aligned} \left| \int \Psi d\mu - \int \Psi d\nu \right| & \leq \sum_{k=0}^N \frac{L_k}{2^k} \alpha_{\mu, \nu}(I_k) \mu(I_k) + \sum_{k=0}^N \left(\frac{1}{\nu(I_{k+1})} \int \Psi_{k+1} d\nu \right) \Delta_{\mu, \nu}(I_k) \mu(I_k) \\ & \quad + \mu(I_{N+1}) \left| \frac{1}{\mu(I_{N+1})} \int \Psi_{N+1} d\mu - \frac{1}{\nu(I_{N+1})} \int \Psi_{N+1} d\nu \right|. \end{aligned} \quad (2.7)$$

This gives (2.5) immediately, observing that $\|\Psi_{N+1}\|_\infty \leq \|\Psi\|_\infty$.

Now, it is time to specify the functions ψ_j . I first define a hands-on Whitney decomposition for $(0, 1/2)$. Pick a small parameter $\tau > 0$, to be specified later, and let $U_0 := [\tau, 1/2 - \tau)$. Then, set $U_{-k} := [\tau 2^{-k}, \tau 2^{-k+1})$ and $U_k := 1/2 - U_{-k}$ for $k \geq 1$. Let $\{\psi_k\}_{k \in \mathbb{Z}}$ be a partition of unity subordinate to slightly enlarged versions of the sets U_k , $k \in \mathbb{Z}$. By this, I first mean that each ψ_k is non-negative and L_k -Lipschitz with

$$L_k \leq \frac{C 2^{|k|}}{\tau}. \quad (2.8)$$

Second, the supports of the functions ψ_k should satisfy $\psi_0 \subset [\tau/2, 1/2 - \tau/2)$,

$$\text{spt } \psi_{-k} \subset [(\tau/2) 2^{-k}, 2\tau 2^{-k+1}) \subset (0, 2\tau 2^{-k+1}) \quad \text{and} \quad \psi_k \subset 1/2 - (0, 2\tau 2^{-k+1})$$

for $k \geq 1$. Third,

$$\sum_{k \in \mathbb{Z}} \psi_k = \chi_{(0, 1/2)}.$$

Let $\Psi^- := \sum_{k > 0} \psi_{-k} + \psi_0/2$ and $\Psi^+ := \sum_{k > 0} \psi_k + \psi_0/2$. Then

$$\Delta_{\mu, \nu}([0, 1]) \leq \left| \int \Psi^- d\mu - \int \Psi^- d\nu \right| + \left| \int \Psi^+ d\mu - \int \Psi^+ d\nu \right|. \quad (2.9)$$

This is the only place in the paper, where the assumption of μ not charging the boundaries of dyadic intervals is used (however, the estimate (2.9) will eventually be applied to all the measures $\mu_I, I \in \mathcal{D}$, so the full strength of the hypothesis is needed). The function Ψ^- is precisely of the form treated above with $I_j := [0, 2^{-j})$, since clearly $\text{spt } \psi_{-k} \subset I_k$. Applying the inequality (2.5) with any $N_1 \in \{0, 1, \dots, \infty\}$ yields

$$\begin{aligned} \left| \int \Psi^- d\mu - \int \Psi^- d\nu \right| &\leq \sum_{k=0}^{N_1} \frac{L_{-k}}{2^k} \alpha_{\mu, \nu}(I_k) \mu(I_k) \\ &\quad + \sum_{k=0}^{N_1} \left(\frac{1}{\nu(I_{k+1})} \int \Psi_{k+1}^- d\nu \right) \Delta_{\mu, \nu}(I_k) \mu(I_k) + 2\mu(I_{N_1+1}). \end{aligned} \quad (2.10)$$

Next, observe that each function $\Psi_{k+1}^-, k \geq 0$, is bounded by 1 and vanishes outside

$$\bigcup_{j=k+1}^{\infty} \text{spt } \psi_{-k} \subset (0, 2\tau 2^{-k}).$$

It follows that

$$\frac{1}{\nu(I_{k+1})} \int \Psi_{k+1}^- d\nu \leq \frac{\nu((0, 2\tau 2^{-k}))}{\nu(I_{k+1})} = o_{D_\nu}(\tau),$$

where the implicit constants only depend on the dyadic doubling constant D_ν of ν . In the sequel, I assume that τ is so small that $o_{D_\nu}(\tau) \leq \kappa$, where $\kappa > 0$ is another small constant, which will eventually depend on the (\mathcal{T}, D) -doubling constant D for μ . Recalling also (2.8), the estimate (2.10) then becomes

$$\left| \int \Psi^- d\mu - \int \Psi^- d\nu \right| \leq \frac{C}{\tau} \sum_{k=0}^{N_1} \alpha_{\mu, \nu}(I_k) \mu(I_k) + \kappa \sum_{k=0}^{N_1} \Delta_{\mu, \nu}(I_k) \mu(I_k) + 2\mu(I_{N_1+1}). \quad (2.11)$$

The last term simply vanishes, if $N_1 = \infty$, because $\mu(\{0\}) = 0$. A heuristic point to observe is that the left hand side is roughly $\Delta_{\mu, \nu}([0, 1])$; the right hand side also contains the same term, but multiplied by a small constant $\kappa > 0$. This gain is "paid for" by the large constant C/τ .

Next, the estimate is replicated for Ψ^+ . This time, the inequality (2.5) is applied to the sequence $\tilde{I}_0 = [0, 1)$, $\tilde{I}_1 = [0, 1/2)$, $\tilde{I}_2 = (\tilde{I}_1)_+$, and in general $\tilde{I}_{k+1} = (\tilde{I}_k)_+$ for $k \geq 1$ (here J_+ is the right half of J). Then, if τ is small enough, it is again clear that $\text{spt } \psi_k \subset \tilde{I}_k$. Thus, by inequality (2.5),

$$\begin{aligned} \left| \int \Psi^+ d\mu - \int \Psi^+ d\nu \right| &\leq \sum_{k=0}^{N_2} \frac{L_k}{2^k} \alpha_{\mu, \nu}(\tilde{I}_k) \mu(\tilde{I}_k) \\ &\quad + \sum_{k=0}^{N_2} \left(\frac{1}{\nu(\tilde{I}_{k+1})} \int \Psi_{k+1}^+ d\nu \right) \Delta_{\mu, \nu}(\tilde{I}_k) \mu(\tilde{I}_k) + 2\mu(\tilde{I}_{N_2+1}) \end{aligned} \quad (2.12)$$

for any $N_2 \geq 0$. As before, the term $\mu(\tilde{I}_{N_2})$ vanishes for $N_2 = \infty$ (because $\mu(\{\frac{1}{2}\}) = 0$), and one can ensure

$$\frac{1}{\nu(\tilde{I}_{k+1})} \int \Psi_{k+1}^+ d\nu \leq \kappa$$

by choosing $\tau = \tau(D_\nu) > 0$ small enough. Consequently (recalling (2.9)), (2.11) and (2.12) together imply

$$\Delta_{\mu,\nu}([0, 1)) \leq \frac{C}{\tau} \sum_{I \in \mathbf{Tail}} \alpha_{\mu,\nu}(I) \mu(I) + \kappa \sum_{I \in \mathbf{Tail}} \Delta_{\mu,\nu}(I) \mu(I) + 2\mu(I_{N_1+1}) + 2\mu(\tilde{I}_{N_2+1}). \quad (2.13)$$

Here \mathbf{Tail} is the collection of all the intervals I_0, \dots, I_{N_1} and $\tilde{I}_0, \dots, \tilde{I}_{N_2}$. The intervals $[0, 1)$ and $[0, 1/2)$ arise a total of two times from (2.11) and (2.12), but this has no visible impact on the end result, (2.13). The estimate (2.13) generalises in a simple way to other

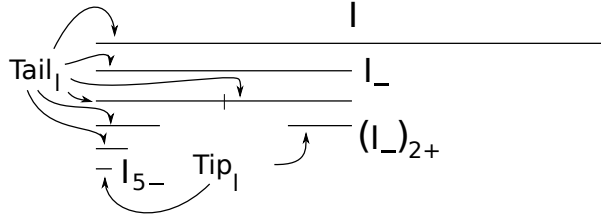


FIGURE 1. An example of $\mathbf{Tail}_I(4, 1)$ and \mathbf{Tip}_I .

intervals $I \in \mathcal{D}$, besides $I = [0, 1)$, but requires an additional piece of notation. Let $I \in \mathcal{D}$, and write $I_{0-} := I =: I_{0+}$. For $k \geq 1$, define $I_{k-} := (I_{(k-1)-})_-$ and $I_{k+} := (I_{(k-1)+})_+$. Now, for a fixed dyadic interval $I \subset [0, 1)$, and $N_1, N_2 \geq 0$, let $\mathbf{Tail}_I = \mathbf{Tail}_I(N_1, N_2)$ be the collection of subintervals of I , which includes I_{k-} for all $0 \leq k \leq N_1$ and $(I_-)_{k+}$ for all $0 \leq k \leq N_2$, see Figure 1. Then, the generalisation of (2.13) reads

$$\Delta_{\mu,\nu}(I) \mu(I) \leq \frac{C}{\tau} \sum_{J \in \mathbf{Tail}_I} \alpha_{\mu,\nu}(J) \mu(J) + \kappa \sum_{J \in \mathbf{Tail}_I} \Delta_{\mu,\nu}(J) \mu(J) + 2\mu(\mathbf{Tip}_I), \quad (2.14)$$

where $\mathbf{Tip}_I = I_{(N_1+1)-} \cup (I_-)_{(N_2+1)+}$. If $N_1 < \infty$ and $N_2 = \infty$, for instance, then $\mathbf{Tip}_I = I_{(N_1+1)-}$. The proof is nothing but an application of (2.13) to the measures μ_I and ν_I . For minor technical reasons, I also wish to allow the choice $N_1 = 0$ and $N_2 = -1$: by definition, this choice means that $\mathbf{Tail}_I = \{I\}$ and $\mathbf{Tip}_I := I_-$. It is easy to see that (2.14) remains valid in this case, with "2" replaced by "4" (for $I = [0, 1)$, this follows by applying (2.11) and (2.12) with the choices $N_1 = 0 = N_2$).

Now, the table is set to prove Proposition 2.4, which I recall here:

Proposition 2.15. *Let μ, ν be measures satisfying the assumptions of the section, and let $\mathcal{T} \subset \mathcal{D}$ be a tree. Moreover, assume that μ is (\mathcal{T}, D) -doubling for some constant $D \geq 1$. Then*

$$\sum_{I \in \mathcal{T}} \Delta_{\mu,\nu}^2(I) \mu(I) \lesssim_{D,\nu} \sum_{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})} \alpha_{\mu,\nu}^2(I) \mu(I) + \mu(\mathbf{Top}(\mathcal{T})).$$

Proof. The sum over $I \in \mathbf{Leaves}(\mathcal{T})$ is evidently bounded by $4\mu(\mathbf{Top}(\mathcal{T}))$, so it suffices to consider

$$I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T}) =: \mathcal{T}^-.$$

Let $I \in \mathcal{T}$, and define the number $N_1 = N_1(I) \geq 0$ as the smallest index so that $I_{(N_1+1)-} \in \mathbf{Leaves}(\mathcal{T})$. If no such index exists, set $N_1 = \infty$. If $I_- \in \mathbf{Leaves}(\mathcal{T})$, then $N_1 = 0$, and I define $N_2 = -1$: then $\mathbf{Tail}_I := \{I\}$, and $\mathbf{Tip}_I := I_-$. Otherwise, if $I_- \in \mathcal{T}^-$, let $N_2 \geq 0$ be the smallest index such that $(I_-)_{(N_2+1)+} \in \mathbf{Leaves}(\mathcal{T})$. If no such index exists, let

$N_2 = \infty$. Now $\mathbf{Tail}_I \subset \mathcal{T}^-$ and $\mathbf{Tip}_I \subset \mathbf{Leaves}(\mathcal{T})$ are defined as after (2.14). Start by the following combination of (2.14) and Cauchy-Schwarz:

$$\begin{aligned} \Delta_{\mu,\nu}^2(I)\mu(I)^2 &\lesssim \frac{1}{\tau^2} \left(\sum_{J \in \mathbf{Tail}_I} \alpha_{\mu,\nu}^2(J)\mu(J)^{3/2} \right) \left(\sum_{J \in \mathbf{Tail}_I} \mu(J)^{1/2} \right) \\ &\quad + \kappa^2 \left(\sum_{J \in \mathbf{Tail}_I} \Delta_{\mu,\nu}^2(J)\mu(J)^{3/2} \right) \left(\sum_{J \in \mathbf{Tail}_I} \mu(J)^{1/2} \right) + \mu(\mathbf{Tip}_I)^2. \end{aligned} \quad (2.16)$$

The factors $\sum_{J \in \mathbf{Tail}_I} \mu(J)^{1/2}$ are under control, thanks to the (\mathcal{T}, D) -doubling hypothesis on μ , and the fact that $\mathbf{Tail}_I \subset \mathcal{T}$. Since \mathbf{Tail}_I consists of two "branches" of nested intervals inside I , and the (\mathcal{T}, D) -doubling hypothesis implies that the μ -measures of intervals decay geometrically along these branches, one arrives at

$$\sum_{J \in \mathbf{Tail}_I} \mu(J)^{1/2} \lesssim_D \mu(I)^{1/2}.$$

Thus, by (2.16),

$$\Delta_{\mu,\nu}^2(I)\mu(I) \lesssim_D \frac{1}{\tau^2} \sum_{J \in \mathbf{Tail}_I} \alpha_{\mu,\nu}^2(J) \frac{\mu(J)^{3/2}}{\mu(I)^{1/2}} + \kappa^2 \sum_{J \in \mathbf{Tail}_I} \Delta_{\mu,\nu}^2(J) \frac{\mu(J)^{3/2}}{\mu(I)^{1/2}} + \frac{\mu(\mathbf{Tip}_I)^2}{\mu(I)}. \quad (2.17)$$

The constant $\kappa > 0$ will have to be chosen so small, eventually, that its product with the implicit constants above is notably less than one. From now on, the precise restriction $J \in \mathbf{Tail}_I$ can be replaced by the conditions $J \in \mathcal{T}^-$ and $J \subset I$. With this in mind, observe first that

$$\begin{aligned} \sum_{I \in \mathcal{T}^-} \sum_{\substack{J \in \mathcal{T}^- \\ J \subset I}} \alpha_{\mu,\nu}^2(J) \frac{\mu(J)^{3/2}}{\mu(I)^{1/2}} &= \sum_{J \in \mathcal{T}^-} \alpha_{\mu,\nu}^2(J) \mu(J)^{3/2} \sum_{\substack{I \in \mathcal{T}^- \\ I \supset J}} \frac{1}{\mu(I)^{1/2}} \\ &\lesssim_D \sum_{J \in \mathcal{T}^-} \alpha_{\mu,\nu}^2(J) \mu(J). \end{aligned}$$

The final inequality uses, again, the geometric decay of μ -measures of intervals in \mathcal{T} . A similar estimate can be performed for the second term in (2.17). As for the third term,

$$\begin{aligned} \sum_{I \in \mathcal{T}^-} \frac{\mu(\mathbf{Tip}_I)^2}{\mu(I)} &\lesssim \sum_{I \in \mathcal{T}^-} \frac{\mu(I_{(N_1+1)-})^2 + \mu((I_-)_{(N_2+1)+})^2}{\mu(I)} \\ &\lesssim \sum_{J \in \mathbf{Leaves}(\mathcal{T})} \mu(J)^2 \sum_{\substack{I \in \mathcal{T}^- \\ I \supset J}} \frac{1}{\mu(I)} \lesssim_D \mu(\mathbf{Leaves}(\mathcal{T})), \end{aligned}$$

relying once more on the geometric decay of μ in \mathcal{T} . Combining all the estimates gives

$$\sum_{I \in \mathcal{T}^-} \Delta_{\mu,\nu}^2(I)\mu(I) \lesssim_D \frac{1}{\tau^2} \sum_{I \in \mathcal{T}^-} \alpha_{\mu,\nu}^2(I)\mu(I) + \kappa^2 \sum_{I \in \mathcal{T}^-} \Delta_{\mu,\nu}^2(I)\mu(I) + \mu(\mathbf{Leaves}(\mathcal{T})). \quad (2.18)$$

If the left hand side is *a priori* finite, the proof of Proposition 2.4 is now completed by choosing κ small enough, depending on D . If not, consider any finite sub-tree $\mathcal{T}_j \subset \mathcal{T}$

with $\mathbf{Top}(\mathcal{T}_j) = \mathbf{Top}(\mathcal{T})$. Then, the proof above gives (2.18) with \mathcal{T}_j in place of \mathcal{T} . Hence

$$\sum_{I \in \mathcal{T}_j^-} \Delta_{\mu, \nu}^2(I) \mu(I) \lesssim_D \sum_{I \in \mathcal{T}_j^-} \alpha_{\mu, \nu}^2(I) \mu(I) + \mu(\mathbf{Top}(\mathcal{T})),$$

where the constants do not depend on the choice of \mathcal{T}_j . Now the proposition follows by letting $\mathcal{T}_j \nearrow \mathcal{T}$. \square

3. ABSOLUTE CONTINUITY OF TREE-ADAPTED MEASURES

Recall the concepts of tree, leaves and boundaries from Definition 2.2, and the notion of (\mathcal{T}, D) -doubling measures from Definition 2.3. In the present section, I assume that $\mathcal{T} \subset \mathcal{D}$ is a tree, and μ, ν are two finite Borel measures, which satisfy the following two assumptions:

- (A) $\min\{\mu(\mathbf{Top}(\mathcal{T})), \nu(\mathbf{Top}(\mathcal{T}))\} > 0$, and
- (B) μ, ν are (\mathcal{T}, D) -doubling for some constant $D \geq 1$.

In particular, the assumptions imply that

$$\mu(I) > 0 \quad \text{and} \quad \nu(I) > 0, \quad I \in \mathcal{T}.$$

For reasons to become apparent soon, I define the (\mathcal{T}, μ) -adaptation of ν ,

$$\nu_{\mathcal{T}} := \nu|_{\partial\mathcal{T}} + \sum_{I \in \mathbf{Leaves}(\mathcal{T})} \frac{\nu}{\mu}(I) \cdot \mu|_I,$$

where $\frac{\nu}{\mu}(I) := \nu(I)/\mu(I)$. Note that

$$\nu_{\mathcal{T}}(I) = \nu(I), \quad I \in \mathcal{T}, \tag{3.1}$$

because $\partial\mathcal{T}$ is disjoint from the leaves, which are also pairwise disjoint. In particular, $\nu_{\mathcal{T}}(\mathbf{Top}(\mathcal{T})) = \nu(\mathbf{Top}(\mathcal{T}))$. The main result of the section is the following:

Proposition 3.2. *Assume (A) and (B), and that*

$$\sum_{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})} \Delta_{\mu, \nu}^2(I) \mu(I) < \infty.$$

Then $\mu|_{\mathbf{Top}(\mathcal{T})} \ll \nu_{\mathcal{T}}$. In particular $\mu|_{\partial\mathcal{T}} \ll \nu$.

Remark 3.3. By the definition of $\nu_{\mathcal{T}}$, it is obvious that $\mu|_{\mathbf{Leaves}(\mathcal{T})} \ll \nu_{\mathcal{T}}$. So, the main point of Proposition 3.2 is to show that $\mu|_{\partial\mathcal{T}} \ll (\nu_{\mathcal{T}})|_{\partial\mathcal{T}} = \nu|_{\partial\mathcal{T}}$.

Since $\mu(\mathbf{Top}(\mathcal{T})) > 0$ and $\nu(\mathbf{Top}(\mathcal{T})) > 0$, one may assume without loss of generality that

$$\mu(\mathbf{Top}(\mathcal{T})) = 1 = \nu(\mathbf{Top}(\mathcal{T})).$$

The proof of Proposition 3.2 is based on a "product representation" for $\nu_{\mathcal{T}}$, relative to μ , in the spirit of [4, Theorem 3.22] of Fefferman, Kenig and Pipher. Recall that every interval $I \in \mathcal{D}$ has exactly two children: I_- and I_+ . Define the μ -adapted Haar functions

$$h_I^{\mu} := c_I^+ \chi_{I_+} - c_I^- \chi_{I_-}, \quad I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T}),$$

where

$$c_I^+ := \frac{\mu(I)}{\mu(I_+)} \quad \text{and} \quad c_I^- := \frac{\mu(I)}{\mu(I_-)}.$$

This ensures that $\int h_I^\mu d\mu = 0$ for $I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$. Note that $\mu(I_+), \mu(I_-) > 0$, because $I_+, I_- \in \mathcal{T}$. Now, the plan is to define coefficients $a_J \in \mathbb{R}$, for $J \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$, so that the following requirement is met:

$$\prod_{\substack{J \supsetneq I \\ J \in \mathcal{T}}} (1 + a_J h_J^\mu)(x) = \frac{\nu}{\mu}(I), \quad x \in I \in \mathcal{T}. \quad (3.4)$$

The left hand side of (3.4) is certainly constant on I , so the equation has some hope; if $I = \mathbf{Top}(\mathcal{T})$, then the product is empty, and the right hand side of (3.4) equals 1 by the assumption $\mu(\mathbf{Top}(\mathcal{T})) = \nu(\mathbf{Top}(\mathcal{T})) = 1$. Now, assume that (3.4) holds for some interval $I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$. Then $I_-, I_+ \in \mathcal{T}$, so if (3.4) is supposed to hold for I_- , one has

$$\frac{\nu}{\mu}(I_-) = \prod_{\substack{J \supsetneq I_- \\ J \in \mathcal{T}}} (1 + a_J h_J^\mu) = (1 - c_I^- a_I) \prod_{\substack{J \supsetneq I \\ J \in \mathcal{T}}} (1 + a_J h_J^\mu) = (1 - c_I^- a_I) \frac{\nu}{\mu}(I), \quad (3.5)$$

and similarly

$$\frac{\nu}{\mu}(I_+) = (1 + c_I^+ a_I) \frac{\nu}{\mu}(I). \quad (3.6)$$

From (3.5) one solves

$$a_I = \frac{\frac{\nu}{\mu}(I) - \frac{\nu}{\mu}(I_-)}{\frac{\nu}{\mu}(I) c_I^-} = \frac{\mu(I_-)}{\mu(I)} - \frac{\nu(I_-)}{\nu(I)}, \quad (3.7)$$

and (3.6) gives

$$a_I = \frac{\frac{\nu}{\mu}(I_+) - \frac{\nu}{\mu}(I)}{\frac{\nu}{\mu}(I) c_I^+} = \frac{\nu(I_+)}{\nu(I)} - \frac{\mu(I_+)}{\mu(I)}. \quad (3.8)$$

Using that $\mu(I_-)/\mu(I) = 1 - \mu(I_+)/\mu(I)$ (and three other similar formulae), it is easy to see that the numbers on the right hand sides of (3.7) and (3.8) agree. So, a_I can be defined consistently, and (3.4) holds for $I_+, I_- \in \mathcal{T}$. Moreover, the formulae for a_I look quite familiar:

Observation 1. $|a_I| = \Delta_{\mu, \nu}(I)$ for $I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$.

Now that the coefficients a_I have been successfully defined for $I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$, let g be the (at the moment) formal series

$$g(x) := \sum_{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})} a_I h_I^\mu(x).$$

Since the Haar functions h_I^μ are orthogonal in $L^2(\mu)$, and satisfy

$$\int (h_I^\mu)^2 d\mu \leq \max\{c_I^+, c_I^-\}^2 \mu(I) \leq D^2 \mu(I), \quad I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T}),$$

one arrives at

$$\|g\|_{L^2(\mu)}^2 = \sum_{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})} \Delta_{\mu, \nu}^2(I) \|h_I\|_{L^2(\mu)}^2 \leq D^2 \sum_{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})} \Delta_{\mu, \nu}^2(I) \mu(I) < \infty,$$

by the assumption in Proposition 3.2. This means that the sequence

$$g_N := \sum_{\substack{I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T}) \\ |I| > 2^{-N}}} a_I h_I^\mu$$

converges in $L^2(\mu)$. In particular, one can pick a subsequence $(g_{N_j})_{j \in \mathbb{N}}$, which converges pointwise μ almost everywhere (in fact, the entire sequence converges by basic martingale theory, but this is not needed). Now, recall that the goal was to prove that $\mu|_{\text{Top}(\mathcal{T})} \ll \nu_{\mathcal{T}}$. To this end, one has to verify that

$$\liminf_{I \rightarrow x} \frac{\mu}{\nu_{\mathcal{T}}}(I) < \infty \quad (3.9)$$

at μ almost every $x \in \text{Top}(\mathcal{T})$. This is clear for $x \in \text{Leaves}(\mathcal{T})$, since the ratios $\mu(I)/\nu_{\mathcal{T}}(I)$, $I \ni x$, are eventually constant. So, it suffices to prove (3.9) at μ almost every point $x \in \partial\mathcal{T}$. Fix a point $x \in \partial\mathcal{T}$ with the properties that sequence $(g_{N_j}(x))_{j \in \mathbb{N}}$ converges, and also

$$\sum_{x \in J \in \mathcal{T}} a_J^2 = \sum_{x \in J \in \mathcal{T}} \Delta_{\mu, \nu}^2(I) < \infty. \quad (3.10)$$

These properties hold at μ almost every $x \in \partial\mathcal{T}$. Let $I \in \mathcal{D}$ be so small that $x \in I \in \mathcal{T}$, and note that

$$\log \frac{\nu_{\mathcal{T}}}{\mu}(I) = \log \frac{\nu}{\mu}(I) = \log \prod_{\substack{J \supseteq I \\ J \in \mathcal{T}}} (1 + a_J h_J^\mu(x)) = \sum_{\substack{J \supseteq I \\ J \in \mathcal{T}}} \log(1 + a_J h_J^\mu(x)).$$

Now, the plan is to use the estimate $\log(1+t) \geq t - C_\delta t^2$, valid as long as $t \geq \delta - 1$ for some $\delta > 0$. Observe that $a_J h_J^\mu(x) \in \{-c_J^- a_J, c_J^+ a_J\}$, where

$$-a_J c_J^- = \frac{\frac{\nu}{\mu}(J_-)}{\frac{\nu}{\mu}(J)} - 1 \geq \frac{1}{C} - 1 \quad \text{and} \quad a_J c_J^+ = \frac{\frac{\nu}{\mu}(J_+)}{\frac{\nu}{\mu}(J)} - 1 \geq \frac{1}{C} - 1.$$

Consequently, for $x \in I \in \mathcal{T}$ with $|I| = 2^{-N_j}$, one has

$$\log \frac{\nu_{\mathcal{T}}}{\mu}(I) \geq \sum_{\substack{J \supseteq I \\ J \in \mathcal{T}}} a_J h_J^\mu(x) - \sum_{\substack{J \supseteq I \\ J \in \mathcal{T}}} (a_J h_J^\mu(x))^2 \geq g_{N_j}(x) - D^2 \sum_{x \in J \in \mathcal{T}} a_J^2.$$

Since the sequence $(g_{N_j}(x))_{j \in \mathbb{N}}$ converges and (3.10) holds, the right hand side has a uniform lower bound $-M(x) > -\infty$. This implies that

$$\limsup_{I \rightarrow x} \frac{\nu_{\mathcal{T}}}{\mu}(I) \geq \exp(-M(x)) > 0,$$

which gives (3.9) at x . The proof of Proposition 3.2 is complete.

4. PROOF OF THE MAIN THEOREM FOR THE DYADIC SQUARE FUNCTION

In this section, Theorem 1.6 is proved via a simple tree construction, coupled with Propositions 2.4 and 3.2 from the previous sections. Recall the statement of Theorem 1.6:

Theorem 4.1. *Assume that μ, ν are Borel probability measures on $[0, 1]$. Assume that μ does not charge the boundaries of dyadic intervals, and ν is dyadically doubling. Then, $\mu|_G \ll \nu$, where*

$$G := \{x \in [0, 1) : \mathcal{S}_\nu(\mu)(x) < \infty\},$$

and \mathcal{S} is the square function

$$\mathcal{S}_{\mathcal{D}, \nu}^2(\mu) = \sum_{I \in \mathcal{D}} \alpha_{\mu, \nu}^2(I) \chi_I.$$

For the rest of the section, fix the measures μ, ν as in the statement above, and let D be the doubling constant of ν . I record a simple lemma, which says that the doubling of ν implies the doubling of μ on intervals, where the α -number is small enough.

Lemma 4.2. *There are constants $\epsilon > 0$ and $C \geq 1$, depending only on D , such that the following holds. For every interval $I \in \mathcal{D}$, if $\alpha_{\mu, \nu}(I) < \epsilon$, then*

$$\mu(I) \leq C \min\{\mu(I_-), \mu(I_+)\}. \quad (4.3)$$

Proof. Let $I_{--} \subset I_-$ and $I_{++} \subset I_+$ be intervals, which lie at distance $\geq |I|/8$ from the boundaries of I_- and I_+ , respectively, and have length $|I|/8$. Let ψ_- and $\psi_+ : \mathbb{R} \rightarrow [0, 1]$ be $(C'/|I|)$ -Lipschitz functions, which equal 1 on I_{--} and I_{++} , respectively, and are supported on I_- and I_+ . Then

$$\frac{\mu(I_-)}{\mu(I)} \geq \frac{1}{\mu(I)} \int \psi_- d\mu \geq \frac{1}{\nu(I)} \int \psi_- d\nu - C' \alpha_{\mu, \nu}(I) \geq \frac{\nu(I_{--})}{\nu(I)} - C' \alpha_{\mu, \nu}(I),$$

and the analogous inequality holds for $\mu(I_+)/\mu(I)$. The ratio $\nu(I_{--})/\nu(I)$ is at least $1/D^3$, so if $\alpha_{\mu, \nu}(I) < 1/(2C'D^3) =: \epsilon$, then both $\mu(I_-) \geq [1/(2D^3)]\mu(I)$ and $\mu(I_+) \geq [1/(2D^3)]\mu(I)$. This gives (4.3) with $C = 2D^3$. \square

In particular, if \mathcal{T} is a tree, and $\alpha_{\mu, \nu}(I) < \epsilon$ for all $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$, then μ is (\mathcal{T}, C) -doubling. I will now describe, how such trees $\mathcal{T}_j \subset \mathcal{D}$ are constructed, starting with \mathcal{T}_0 . Let $[0, 1) = \text{Top}(\mathcal{T}_0)$, and assume that some interval $I \in \mathcal{T}_0$. If

$$\sum_{I \subset J \subset [0, 1)} \alpha_{\mu, \nu}^2(J) \geq \epsilon^2, \quad (4.4)$$

add I to $\text{Leaves}(\mathcal{T}_0)$. The children I_- and I_+ become the tops of new trees. If (4.4) fails, add I_- and I_+ to \mathcal{T}_0 . The construction of \mathcal{T}_0 is now complete. If a new top T_j was created in the process of constructing \mathcal{T}_0 , and $\mu(T_j) > 0$, construct a new tree \mathcal{T}_j with $\text{Top}(\mathcal{T}_j) = T_j$ by repeating the algorithm above, only replacing $[0, 1)$ by T_j in the stopping criterion (4.4). Continue this process until all intervals in \mathcal{D} belong to some tree, or all remaining tops T_j satisfy $\mu(T_j) = 0$. For all tops T_j with $\mu(T_j) = 0$, simply define $\mathcal{T}_j := \{I \in \mathcal{D} : I \subset T_j\}$, so there is no further stopping inside \mathcal{T}_j .

Remark 4.5. Let \mathcal{T} be one of the trees constructed above, with $\mu(\text{Top}(\mathcal{T})) > 0$. Then μ is (\mathcal{T}, C) -doubling by Lemma 4.2, since it is clear that $\alpha_{\mu, \nu}(I) < \epsilon$ for all $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$. In particular $\mu(I) > 0$ for all $I \in \mathcal{T}$.

The following observation is now rather immediate from the definitions:

Lemma 4.6. *Assume that $\mathcal{T}_0, \dots, \mathcal{T}_{N-1}$ are distinct trees such that $x \in \text{Leaves}(\mathcal{T}_j)$ for all $0 \leq j \leq N-1$. Then*

$$\mathcal{S}_{\mathcal{D}, \nu}^2(\mu)(x) \geq \epsilon^2 N.$$

Proof. For $0 \leq j \leq N-1$, Let $I_j \in \text{Leaves}(\mathcal{T}_j)$ with $x \in I_j$. Then

$$\mathcal{S}_{\mathcal{D}, \nu}^2(\mu)(x) \geq \sum_{j=0}^{N-1} \sum_{I_j \subset J \subset \text{Top}(\mathcal{T}_j)} \alpha_{\mu, \nu}^2(J) \geq \epsilon^2 N,$$

as claimed. \square

It follows that μ almost every point in $G = \{x \in [0, 1] : \mathcal{S}_\nu(\mu)(x) < \infty\}$ belongs to $\text{Leaves}(\mathcal{T}_j)$ for only finitely many trees \mathcal{T}_j . This is equivalent to saying that μ almost every point in G belongs to $\partial\mathcal{T}$ for some tree \mathcal{T} . The converse is also true: if x belongs to $\partial\mathcal{T}$ for some tree \mathcal{T} , then clearly $\mathcal{S}_\nu(\mu)(x) < \infty$. Consequently

$$\mu|_G = \sum_{\text{trees } \mathcal{T}} \mu|_{\partial\mathcal{T}}.$$

To prove Theorem 4.1, it now suffices to show that $\mu|_{\partial\mathcal{T}} \ll \nu$ for every tree \mathcal{T} . This is clear, if $\mu(\text{Top}(\mathcal{T})) = 0$, so I exclude the trivial case to begin with. In the opposite case, note that

$$\sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \alpha_{\mu, \nu}^2(I) \mu(I) = \int \sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \alpha_{\mu, \nu}^2(I) \chi_I(x) d\mu x \leq \epsilon^2 \cdot \mu(\text{Top}(\mathcal{T})). \quad (4.7)$$

It then follows from Proposition 2.4 that

$$\sum_{I \in \mathcal{T}} \Delta_{\mu, \nu}^2(I) \mu(I) \lesssim \mu(\text{Top}(\mathcal{T})) < \infty,$$

and the claim $\mu|_{\partial\mathcal{T}} \ll \nu$ is finally a consequence of Proposition 3.2. The proof of Theorem 1.6 is complete.

5. THE NON-DYADIC SQUARE FUNCTION

This section contains the proof of Theorem 1.7. The argument naturally contains many similarities to the one given above. The main novelty is that one needs to work with "smooth" versions of the α -numbers, which essentially coincide with the one considered by Azzam, David and Toro in [2, Section 5].

5.1. Smooth α -numbers, and their properties. Here is the definition:

Definition 5.1 (Smooth α -numbers). Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the 1-Lipschitz function

$$\varphi = \chi_{[0,1]} \text{dist}(\cdot, \{0, 1\}).$$

For an interval $I \subset \mathbb{R}$, define

$$\alpha_{s, \mu, \nu}(I) := \mathbb{W}_1(\mu_{\varphi, I}, \nu_{\varphi, I}),$$

where

$$\mu_{\varphi, I} := \frac{T_{I\#}(\mu|_I)}{\mu(\varphi_I)} \quad \text{and} \quad \nu_{\varphi, I} := \frac{T_{I\#}(\nu|_I)}{\nu(\varphi_I)}.$$

Here $\varphi_I = \varphi \circ T_I$, and $\mu(\varphi_I) = \int \varphi_I d\mu$. If $\mu(\varphi_I) = 0$ (or $\nu(\varphi_I) = 0$), set $\mu_{\varphi, I} \equiv 0$ (or $\nu_{\varphi, I} \equiv 0$). Unwrapping the definition, if $\mu(\varphi_I), \nu(\varphi_I) > 0$, then

$$\alpha_{s, \mu, \nu}(I) = \sup_{\psi} \left| \frac{1}{\mu(\varphi_I)} \int \psi \circ T_I d\mu - \frac{1}{\nu(\varphi_I)} \int \psi \circ T_I d\nu \right| = \sup_{\psi} \left| \frac{\mu(\psi_I)}{\mu(\varphi_I)} - \frac{\nu(\psi_I)}{\nu(\varphi_I)} \right|,$$

where the sup is taken over test functions ψ .

The main reason to prefer the smooth α -numbers over the ones from Definition 1.3 is the following: if $I \subset J$ are intervals of comparable length, then $\alpha_{s, \mu, \nu}(I) \lesssim \alpha_{s, \mu, \nu}(J)$, whenever ν is a globally doubling measure. This fact will only be used somewhat implicitly (with details included on the spot), but I include a proof within the proposition

below (see also [2, Lemma 5.2] for a similar statement). The same relation is not true for the numbers $\alpha_{\mu,\nu}(I)$ and $\alpha_{\mu,\nu}(J)$, even for very nice measures μ and ν , see Example 5.4.

Proposition 5.2 (Basic properties of the smooth α -numbers). *Let μ, ν be two Radon measures on \mathbb{R} , and let $I \subset \mathbb{R}$ be an interval. Then*

$$\alpha_{s,\mu,\nu}(I) \leq 2 \quad \text{and} \quad \alpha_{s,\mu,\nu}(I) \leq \frac{2\alpha_{\mu,\nu}(I)}{\nu_I(\varphi)}.$$

Moreover, if ν is doubling with constant D , the following holds. If $I \subset J \subset \mathbb{R}$ are intervals with $|I| \geq \theta|J|$ for some $\theta > 0$, then

$$\alpha_{s,\mu,\nu}(I) \lesssim_{D,\theta} \alpha_{s,\mu,\nu}(J). \quad (5.3)$$

Proof. For the duration of the proof, fix an interval $I \subset \mathbb{R}$ with $\mu(\varphi_I), \nu(\varphi_I) > 0$. The cases, where $\mu(\varphi_I) = 0$ or $\nu(\varphi_I) = 0$ always require a little case chase, which I omit. Recall that $\varphi = \chi_{[0,1]} \text{dist}(\cdot, \{0, 1\})$. Note that any 1-Lipschitz function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ supported on $[0, 1]$ must satisfy $|\psi| \leq \varphi$. Consequently $|\psi_I| \leq \varphi_I$ for any interval I , and so

$$\alpha_{s,\mu,\nu}(I) \leq \sup_{\psi} \left[\frac{\mu(|\psi_I|)}{\mu(\varphi_I)} + \frac{\nu(|\psi_I|)}{\nu(\varphi_I)} \right] \leq 2.$$

This proves the first inequality. For the second inequality, one may assume that $\alpha_{\mu,\nu}(I) > 0$, since otherwise $\mu|_{\text{int } I} = c\nu|_{\text{int } I}$ for some constant $c > 0$, and this also gives $\alpha_{s,\mu,\nu}(I) = 0$. After this observation, it is easy to reduce to the case $\mu(\varphi_I) > 0$ and $\nu(\varphi_I) > 0$. Fix a test function ψ . Using that $\mu_I(|\psi|) = \mu(|\psi_I|)/\mu(I) \leq \mu(\varphi_I)/\mu(I) = \mu_I(\varphi)$, one obtains

$$\begin{aligned} \left| \frac{\mu(\psi_I)}{\mu(\varphi_I)} - \frac{\nu(\psi_I)}{\nu(\varphi_I)} \right| &= \left| \frac{\mu_I(\psi)}{\mu_I(\varphi)} - \frac{\nu_I(\psi)}{\nu_I(\varphi)} \right| = \left| \frac{\mu_I(\psi)\nu_I(\varphi) - \nu_I(\psi)\mu_I(\varphi)}{\mu_I(\varphi)\nu_I(\varphi)} \right| \\ &\leq \frac{\mu_I(|\psi|)}{\mu_I(\varphi)\nu_I(\varphi)} |\mu_I(\varphi) - \nu_I(\varphi)| + \frac{\mu_I(\varphi)}{\mu_I(\varphi)\nu_I(\varphi)} |\mu_I(\psi) - \nu_I(\psi)| \leq \frac{2\alpha_{\mu,\nu}(I)}{\nu_I(\varphi)}. \end{aligned}$$

To prove the final claim, start with the following estimate for a test function ψ :

$$\left| \frac{\mu(\psi_I)}{\mu(\varphi_I)} - \frac{\nu(\psi_I)}{\nu(\varphi_I)} \right| \leq \frac{\nu(\varphi_J)}{\nu(\varphi_I)} \left| \frac{\mu(\psi_I)}{\mu(\varphi_J)} - \frac{\nu(\psi_I)}{\nu(\varphi_J)} \right| + \frac{\mu(|\psi_I|)}{\mu(\varphi_I)} \frac{\nu(\varphi_J)}{\nu(\varphi_I)} \left| \frac{\mu(\varphi_I)}{\mu(\varphi_J)} - \frac{\nu(\varphi_I)}{\nu(\varphi_J)} \right|.$$

Then, recall that $\mu(|\psi_I|) \leq \mu(\varphi_I)$. Further, it follows from the doubling of ν that $\nu(\varphi_J) \lesssim_{D,\theta} \nu(\varphi_I)$. Finally, notice that $\psi_I = (\psi_I \circ T_J^{-1}) \circ T_J$ and $\varphi_I = (\varphi_I \circ T_J^{-1}) \circ T_J$, where both

$$\psi_I \circ T_J^{-1} \quad \text{and} \quad \varphi_I \circ T_J^{-1}$$

are $(|J|/|I|)$ -Lipschitz functions supported on $T_J(I) \subset [0, 1]$. Consequently,

$$\max \left\{ \left| \frac{\mu(\psi_I)}{\mu(\varphi_J)} - \frac{\nu(\psi_I)}{\nu(\varphi_J)} \right|, \left| \frac{\mu(\varphi_I)}{\mu(\varphi_J)} - \frac{\nu(\varphi_I)}{\nu(\varphi_J)} \right| \right\} \leq \frac{\alpha_{s,\mu,\nu}(J)}{\theta},$$

and the estimate (5.3) follows. \square

The last property in the proposition fails for the numbers $\alpha_{\mu,\nu}(I)$, as the following example demonstrates:

Example 5.4. Fix $n \in \mathbb{N}$, and let $I_-^n := [\frac{1}{2} - 2^{-n}, \frac{1}{2}]$ and $I_+^n := (\frac{1}{2}, \frac{1}{2} + 2^{-n}]$. Let μ be the same measure as in Example 2.1:

$$\mu = \chi_{\mathbb{R} \setminus (I_-^n \cup I_+^n)} + \frac{\chi_{I_-^n}}{2} + \frac{3\chi_{I_+^n}}{2}.$$

Let $\nu = \mathcal{L}$. It is clear that both μ and ν are doubling, with constants independent of n . It is also easy to check that $\alpha_{\mu,\nu}(I) \lesssim 2^{-2n}$ for any interval I with length $|I| \sim 1$ such that $I_-^n \cup I_+^n \subset I$ (this implies that $\mu(I) = \nu(I)$). However, $\alpha_{\mu,\nu}([0, 1/2]) \sim 2^{-n}$, because $\nu_{[0,1/2]} = \chi_{[0,1]}$, while

$$\mu_{[0,1/2]} = \left(1 + \frac{2^{-n}}{1 - 2^{-n}}\right) \chi_{[0,1-2^{1-n})} + \frac{1}{2} \left(1 + \frac{2^{-n}}{1 - 2^{-n}}\right) \chi_{[1-2^{1-n},1]}.$$

So, for instance, it is clear that no inequality of the form $\alpha_{\mu,\nu}([0, 1/2]) \lesssim \alpha_{\mu,\nu}([-1, 1])$ can hold.

Without the doubling assumption on ν , even the smooth α -numbers can behave badly:

Example 5.5. Let $\mu = \delta_{1/2}$, and $\nu = (1 - \epsilon) \cdot \delta_{1/2+\epsilon} + \epsilon \cdot \delta_{1/4}$. Then $\alpha_{s,\mu,\nu}([-1, 1]) \sim \epsilon$, but $\alpha_{s,\mu,\nu}([0, 1/2]) \sim 1$.

5.2. The proof of Theorem 1.7. In this section, ν is a globally doubling measure with constant $D \geq 1$, say. In particular if φ is the special function from the definition of the smooth α -numbers, then

$$\nu_I(\varphi) = \frac{\nu(\varphi_I)}{\nu(I)} \gtrsim_D 1, \quad I \subset \mathbb{R}.$$

It then follows from Proposition 5.2 that

$$\tilde{\mathcal{S}}_\nu^2(\mu)(x) := \int_0^1 \alpha_{s,\mu,\nu}^2(B(x, r)) \frac{dr}{r} \lesssim_D \int_0^1 \alpha_{\mu,\nu}^2(B(x, r)) \frac{dr}{r} = \mathcal{S}_\nu^2(\mu)(x)$$

for all $x \in \text{spt } \mu$. Consequently, to prove Theorem 1.7, it suffices to show that $\mu|_{\tilde{G}} \ll \nu$, where

$$\tilde{G} := \{x : \tilde{\mathcal{S}}_\nu(\mu)(x) < \infty\}.$$

I write

$$\alpha_{s,\mu,\nu}(J) =: \alpha(J), \quad J \subset \mathbb{R}.$$

Assume without loss of generality (or translate both measures μ and ν slightly) that $\mu(\partial I) = 0$ for all $I \in \mathcal{D}$. Also without loss of generality, one may assume that $\text{spt } \mu \subset (0, 1)$: the reason is that the finiteness $\tilde{\mathcal{S}}_\nu(\mu)(x)$ (or $\mathcal{S}_\nu(\mu)(x)$ for that matter) is equivalent to the finiteness of $\tilde{\mathcal{S}}_\nu(\mu|_U)(x)$ for all $x \in U$, whenever $U \subset \mathbb{R}$ is open. So, it suffices to prove $\mu|_{U \cap \tilde{G}} \ll \nu$ for any bounded open set U . Whenever I write \mathcal{D} in the sequel, I only mean the family $\{I \in \mathcal{D} : I \subset [0, 1]\}$.

I start with some standard discretisation arguments. For each $I \in \mathcal{D}$, associate a somewhat larger interval $B_I \supset I$ as follows. First, for $x \in \text{spt } \mu$ and $k \in \mathbb{N}$, choose a radius $r_{x,k} > 0$ such that

$$\alpha(B(x, r_{x,k})) \leq 2 \inf\{\alpha(B(x, r)) : 1.1 \cdot 2^{-k-1} \leq r \leq 0.9 \cdot 2^{-k}\}. \quad (5.6)$$

Then

$$\alpha^2(B(x, r_{x,k})) \leq \left(\frac{1}{\ln[2 \cdot (0.9/1.1)]} \int_{1.1 \cdot 2^{-k-1}}^{0.9 \cdot 2^{-k}} 2\alpha(x, r) \frac{dr}{r} \right)^2 \lesssim \int_{2^{-k-1}}^{2^{-k}} \alpha^2(x, r) \frac{dr}{r}.$$

For $I \in \mathcal{D}$ with $|I| = 2^{-k}$ and $I \cap \text{spt } \mu \neq \emptyset$, let B_I be some open interval of the form $B(x, r_{x,k-10})$, $x \in I$, such that

$$\alpha(B_I) \leq 2 \inf\{\alpha(B(y, r_{y,k-10})) : y \in I \cap \text{spt } \mu\}.$$

The number "10" simply ensures that $I \subset B_I$ with $\text{dist}(I, \partial B_I) \sim |I|$, and

$$I \subset J \implies B_I \subset B_J, \quad \text{for } I, J \in \mathcal{D}.$$

This implication also uses the slight separation between the scales, provided by the factors "1.1" and "0.9" in (5.6). For $I \in \mathcal{D}$ with $I \cap \text{spt } \mu = \emptyset$, define $B_I := I$ (although this definition will never be really used). Now, a tree decomposition of \mathcal{D} can be performed as in the previous section, replacing the stopping condition (4.4) by declaring $\text{Leaves}(\mathcal{T})$ to consist of the maximal intervals $I \subset \text{Top}(\mathcal{T})$ with

$$\sum_{I \subset J \subset \text{Top}(\mathcal{T})} \alpha^2(B_I) \geq \epsilon^2,$$

where $\epsilon = \epsilon_D > 0$ is a suitable small number; in particular, $\epsilon > 0$ is chosen so small that $\alpha(B_I) \leq \epsilon$ implies $\mu(B_I) \lesssim \mu(I)$ (which is possible by a small modification of Lemma 4.2). If now $x \in \text{Leaves}(\mathcal{T})$ for infinitely many different trees \mathcal{T} , then

$$\infty = \sum_{x \in I \in \mathcal{D}} \alpha^2(B_I) \leq 2 \sum_{k \in \mathbb{N}} \alpha^2(B(x, r_{x, k-10})) \lesssim \int_0^{2^{10}} \alpha^2(B(x, r)) \frac{dr}{r},$$

which implies that $x \notin \tilde{G}$. Repeating the argument from Section 4, this gives

$$\mu|_{\tilde{G}} \leq \sum_{\text{trees } \mathcal{T}} \mu|_{\partial \mathcal{T}}.$$

The converse inequality could also be deduced from the stability of the smooth α -numbers (Proposition 5.2), but it is not needed: the inequality already shows that it suffices to prove

$$\mu|_{\partial \mathcal{T}} \ll \nu \tag{5.7}$$

for any given tree \mathcal{T} . So, fix a tree \mathcal{T} . If $\epsilon > 0$ was chosen small enough (again depending on D), then μ is (\mathcal{T}, C) -doubling for some $C = C_D \geq 1$ in the usual sense:

$$\mu(\hat{I}) \leq C\mu(I), \quad I \in \mathcal{T} \setminus \text{Top}(\mathcal{T}).$$

So, if one knew that

$$\sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \Delta_{\mu, \nu}^2(I) \mu(I) < \infty, \tag{5.8}$$

then the familiar Proposition 3.2 would imply (5.7), completing the entire proof.

The proof of (5.8) is based on the following inequality:

$$\sum_{I \in \mathcal{T}} \Delta_{\mu, \nu}^2(I) \mu(I) \lesssim \sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \alpha^2(B_I) \mu(B_I) + \mu(\text{Top}(\mathcal{T})). \tag{5.9}$$

The right hand side is finite by the same estimate as in (4.7) (start with $\mu(B_I) \lesssim \mu(I)$, using $\alpha(B_I) \leq \epsilon$ for $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$). So, (5.9) implies (5.8). I start the proof of (5.9) by noting that if $I \in \mathcal{D}$, then

$$\begin{aligned} \Delta_{\mu, \nu}(I) &= \left| \frac{\nu(I_-)}{\nu(I)} - \frac{\mu(I_-)}{\mu(I)} \right| \\ &\leq \frac{\nu(\varphi_{B_I})}{\nu(I)} \left| \frac{\nu(I_-)}{\nu(\varphi_{B_I})} - \frac{\mu(I_-)}{\mu(\varphi_{B_I})} \right| + \frac{\mu(I_-)}{\mu(I)} \frac{\nu(\varphi_{B_I})}{\nu(I)} \left| \frac{\mu(I)}{\mu(\varphi_{B_I})} - \frac{\nu(I)}{\nu(\varphi_{B_I})} \right|. \end{aligned} \tag{5.10}$$

Noting that $\nu(\varphi_{B_I})/\nu(I) \lesssim_D 1$, to prove (5.9), it suffices to control

$$\sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \left[\left| \frac{\nu(I_-)}{\nu(\varphi_{B_I})} - \frac{\mu(I_-)}{\mu(\varphi_{B_I})} \right|^2 + \left| \frac{\mu(I)}{\mu(\varphi_{B_I})} - \frac{\nu(I)}{\nu(\varphi_{B_I})} \right|^2 \right] \mu(I) \quad (5.11)$$

by the right hand side of (5.9). The main task is to find a suitable replacement for the "Tail – Tip" inequality (2.14), which I replicate here for comparison:

$$\Delta_{\mu,\nu}(I)\mu(I) \leq \frac{C}{\tau} \sum_{J \in \text{Tail}_I} \alpha_{\mu,\nu}(J)\mu(J) + \kappa \sum_{J \in \text{Tail}_I} \Delta_{\mu,\nu}(J)\mu(J) + 2\mu(\text{Tip}_I). \quad (5.12)$$

Glancing at (5.11), one sees that an analogue for the inequality above is actually needed for both the terms

$$\tilde{\Delta}_{B_I}(I_-) = \left| \frac{\nu(I_-)}{\nu(\varphi_{B_I})} - \frac{\mu(I_-)}{\mu(\varphi_{B_I})} \right| \quad \text{and} \quad \tilde{\Delta}_{B_I}(I) = \left| \frac{\mu(I)}{\mu(\varphi_{B_I})} - \frac{\nu(I)}{\nu(\varphi_{B_I})} \right|.$$

If $I_- \in \text{Leaves}(\mathcal{T})$, then the trivial estimate $\tilde{\Delta}_{B_I}(I_-) \lesssim 1$ will suffice, so in the sequel I assume that

$$I, I_- \notin \text{Leaves}(\mathcal{T}). \quad (5.13)$$

The goal is inequality (5.18) below. Fix B_I and $J \in \{I, I_-\}$. Assume for notational convenience that $|B_I| = 1$, and hence, also $|J| \sim 1$. In a familiar manner, start by writing

$$\chi_J = \sum_{k \in \mathbb{Z}} \psi_k, \quad (5.14)$$

where ψ_k is a non-negative $C2^{|k|}$ -Lipschitz function supported on either $J \subset B_I$ (for $k = 0$), or $J_{|k|_-}$ (for negative k) or J_{k+} (for positive k). As in the proof of the original Tail – Tip inequality, it suffices to first estimate

$$\left| \frac{1}{\mu(\varphi_{B_I})} \int \Psi_0^+ d\mu - \frac{1}{\nu(\varphi_{B_I})} \int \Psi_0^+ d\nu \right|, \quad (5.15)$$

where $\Psi_0^+ = \sum_{k \geq 1} \psi_k + \psi_0/2$, and more generally $\Psi_j^+ = \sum_{k \geq j} \psi_k$ for $j \geq 1$; eventually one can just replicate the argument for the function $\Psi_0^- = \sum_{k \leq -1} \psi_k + \psi_0/2$, and summing the bounds gives control for $\tilde{\Delta}_{B_I}(J)$. Start with the following estimate, which only uses the triangle inequality, and the fact that $\psi_0/2$ is a C -Lipschitz function supported on B_I :

$$\begin{aligned} & \left| \frac{1}{\mu(\varphi_{B_I})} \int \Psi_0^+ d\mu - \frac{1}{\nu(\varphi_{B_I})} \int \Psi_0^+ d\nu \right| \leq C\alpha(B_I) \\ & + \frac{\mu(\varphi_{B_{J_+}})}{\mu(\varphi_{B_I})} \left| \frac{1}{\mu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\mu - \frac{1}{\nu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\nu \right| \\ & + \left(\frac{1}{\nu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\nu \right) \left| \frac{\mu(\varphi_{B_{J_+}})}{\mu(\varphi_{B_I})} - \frac{\nu(\varphi_{B_{J_+}})}{\nu(\varphi_{B_I})} \right|. \end{aligned} \quad (5.16)$$

Here

$$\frac{1}{\nu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\nu \lesssim 1,$$

since ν is doubling and Ψ_1^+ vanishes outside $J_+ \subset B_{J_+}$, and

$$\left| \frac{\mu(\varphi_{B_{J_+}})}{\mu(\varphi_{B_I})} - \frac{\nu(\varphi_{B_{J_+}})}{\nu(\varphi_{B_I})} \right| \leq \frac{|B_I|}{|B_{J_+}|} \cdot \alpha(B_I) \lesssim \alpha(B_I),$$

since $\varphi_{B_{J_+}} = (\varphi_{B_{J_+}} \circ T_{B_I}^{-1}) \circ T_{B_I}$, where $\varphi_{B_{J_+}} \circ T_{B_I}^{-1}$ is a $(|B_I|/|B_{J_+}|)$ -Lipschitz function supported on $[0, 1]$. Consequently,

$$\begin{aligned} & \left| \frac{1}{\mu(\varphi_{B_I})} \int \Psi_0^+ d\mu - \frac{1}{\nu(\varphi_{B_I})} \int \Psi_0^+ d\nu \right| \mu(\varphi_{B_I}) \leq C\alpha(B_I)\mu(\varphi_{B_I}) \\ & + \left| \frac{1}{\mu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\mu - \frac{1}{\nu(\varphi_{B_{J_+}})} \int \Psi_1^+ d\nu \right| \mu(\varphi_{B_{J_+}}) \end{aligned}$$

Here Ψ_1^+ vanishes outside on $J_+ \subset B_{J_+}$, so the estimate can be iterated. After $N \geq 0$ repetitions (the case $N = 0$ was seen above), one ends up with

$$\begin{aligned} & \left| \frac{1}{\mu(\varphi_{B_I})} \int \Psi_0^+ d\mu - \frac{1}{\nu(\varphi_{B_I})} \int \Psi_0^+ d\nu \right| \mu(\varphi_{B_I}) \leq C \sum_{k=0}^N \alpha(B_{J_{k+}}) \mu(\varphi_{B_{J_{k+}}}) \\ & + \mu(\varphi_{B_{J_{(N+1)+}}}) \left| \frac{1}{\mu(\varphi_{B_{(N+1)+}})} \int \Psi_{N+1}^+ d\mu - \frac{1}{\nu(B_{(N+1)+})} \int \Psi_{N+1}^+ d\nu \right|, \end{aligned} \quad (5.17)$$

where one needs to interpret $J_{0+} = I$ (which is different from J in case $J = I_-$). What is a good choice for N ? Let $N_1 \geq 0$ be the smallest number such that $J_{(N_1+1)+} \in \mathbf{Leaves}(\mathcal{T})$. If there is no such number, let $N_1 = \infty$. In case $N_1 = \infty$, the term on line (5.17) vanishes, since $\mu(B_{J_{N+}})$ decays rapidly as long as $N \in \mathcal{T}$ (using the doubling of ν , and the fact that $\alpha(B_I) \leq \epsilon$ for $I \in \mathcal{T}$). If $N_1 < \infty$, the term on line (5.17) is clearly bounded by $\leq 2\mu(B_{J_{(N_1+1)+}})$, since Ψ_{N+1}^+ vanishes outside $J_{(N_1+1)+}$, which is well inside $B_{(N_1+1)+}$. Observing that also $\mu(I) \lesssim \mu(\varphi_{B_I})$, it follows that

$$\left| \frac{1}{\mu(\varphi_{B_I})} \int \Psi_0^+ d\mu - \frac{1}{\nu(\varphi_{B_I})} \int \Psi_0^+ d\nu \right| \mu(I) \lesssim \sum_{k=0}^{N_1} \alpha(B_{J_{k+}}) \mu(B_{J_{k+}}) + \mu(B_{J_{(N_1+1)+}}).$$

Finally, by symmetry, the same argument can be carried out for the series $\Psi_0^- = \sum_{k < 0} \psi_k + \psi_0/2$. If $N_2 \geq 0$ is the smallest number such that $J_{(N_2+1)-} \in \mathbf{Leaves}(\mathcal{T})$, this leads to the following analogue of the **Tail – Tip** inequality:

$$\tilde{\Delta}_{B_I}(J)\mu(I) \lesssim \sum_{P \in \mathbf{Tail}_J} \alpha(B_P)\mu(B_P) + \mu(\mathbf{Tip}_J), \quad J \in \{I, I_-\}, I \in \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T}). \quad (5.18)$$

Here \mathbf{Tail}_J is the collection of dyadic intervals $\mathbf{Tail}_J = \{J_{N_2-}, \dots, J, \dots, J_{N_1+}\} \subset \mathcal{T} \setminus \mathbf{Leaves}(\mathcal{T})$, and $\mathbf{Tip}_J = B_{J_{(N_2+1)-}} \cup B_{J_{(N_1+1)+}}$. Finally, in the excluded special case, where $J = I_- \in \mathbf{Leaves}(\mathcal{T})$ (recall (5.13)), the same estimate holds, if one defines $\mathbf{Tail}_J = \emptyset$ and $\mathbf{Tip}_J := J$ (noting that $I \in \mathcal{T}$, so $\mu(I) \lesssim \mu(J)$).

Remark 5.19. The proof of (5.18) is the reason, why the smooth α -numbers were introduced. Once the inequality is proven, it is clear that the right hand side could be further bounded by the "real" α -numbers. Curiously, however, proving the inequality directly

for the "real" α -numbers is difficult: if one used the normalising factors " $\mu(B_I)$ " instead of " $\mu(\varphi_{B_I})$ ", then the "error" term on line (5.16) would look like this:

$$\left| \frac{\mu(B_{J_+})}{\mu(B_I)} - \frac{\nu(B_{J_+})}{\nu(B_I)} \right|.$$

Such a term can neither be controlled by $\alpha_{\mu,\nu}(B_I)$, nor be "absorbed" as in Section 2 (the factor is no longer of the form appearing on the left hand side of (5.9)).

Armed with the **Tail – Tip** inequality (5.18), the proof of the main estimate (5.9) is a replica of the argument in the dyadic case, namely the proof of Proposition 2.4. I only sketch the details. For $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$, and $J \in \{I, I_-\}$, start with

$$\begin{aligned} \tilde{\Delta}_{B_I}^2(J)\mu(I) &\lesssim \sum_{P \in \text{Tail}_J} \alpha^2(B_P) \frac{\mu(B_P)^{3/2}}{\mu(I)^{1/2}} + \frac{\mu(\text{Tip}_J)^2}{\mu(I)} \\ &\leq \sum_{\substack{P \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T}) \\ P \subset I}} \alpha^2(B_P) \frac{\mu(B_P)^{3/2}}{\mu(I)^{1/2}} + \frac{\mu(\text{Tip}_J)^2}{\mu(I)}. \end{aligned}$$

The second inequality is trivial, and the first is proved with the same Cauchy-Schwarz argument as (2.17), using the fact that $\sum_{P \in \text{Tail}_J} \mu(B_P)^{1/2} \lesssim \mu(I)^{1/2}$, which follows from $\text{Tail}_J \subset \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$, and in particular the geometric decay of the measures $\mu(B_P)$ for $P \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$. Now, the inequality above can be summed for $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$ precisely as in the proof of (2.18). In particular, one should first use the estimate

$$\mu(\text{Tip}_J) \leq \mu(B_{J_{(N_2+1)-}}) + \mu(B_{J_{(N_1+1)+}}) \lesssim \mu(J_{(N_2+1)-}) + \mu(J_{(N_1+1)+}),$$

which follows from $\alpha(B_{J_{N_1+}}), \alpha(B_{J_{N_2-}}) < \epsilon$, if ϵ is small enough, depending on the doubling constant of ν . The conclusion is

$$\sum_{I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \tilde{\Delta}_{B_I}^2(J)\mu(I) \lesssim \sum_{P \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})} \alpha^2(B_P)\mu(B_P) + \mu(\text{Leaves}(\mathcal{T}))$$

for $J \in \{I, I_-\}$. As observed in and around (5.11), this implies (5.9).

Remark 5.20. In the proof of (5.9), the uniform bound $\alpha(B_I) < \epsilon$, $I \in \mathcal{T} \setminus \text{Leaves}(\mathcal{T})$, was only used to guarantee that μ is sufficiently doubling along, and inside, the balls B_I . If such properties are assumed *a priori* in some given tree \mathcal{T} , then (5.9) continues to hold for \mathcal{T} . In particular, if μ is doubling on the whole real line, and Carleson condition

$$\int_{B(x, 2r)} \int_0^{2r} \alpha_{\mu,\nu}^2(B(y, t)) \frac{dt d\mu y}{t} \leq C\mu(B(x, r)),$$

holds, then the dyadic Carleson condition of Theorem 1.8 holds for any dyadic system \mathcal{D} (a family of half-open intervals covering \mathbb{R} , where every interval has length of the form 2^{-k} for some $k \in \mathbb{Z}$, and every interval is the union of two further intervals in the family; the proof of Theorem 1.8 seen in Section 2 works for any such system). It follows from this that $\mu \in A_\infty^\mathcal{D}(\nu)$ for every dyadic system \mathcal{D} , and consequently $\mu \in A_\infty(\nu)$. (To see this,

pick a finite collection $\mathcal{D}_1, \dots, \mathcal{D}_N$ of dyadic systems so that the max of the corresponding dyadic maximal functions $M_\nu^{\mathcal{D}_i}$,

$$M_\nu^{\mathcal{D}_i} f(x) = \sup_{x \in I \in \mathcal{D}_i} \frac{1}{\nu(I)} \int_I |f| d\nu,$$

bounds the usual Hardy-Littlewood maximal function M_ν , up to a constant depending only on the doubling of ν . The construction of such systems is well-known, and in \mathbb{R} as few as 2 systems do the trick; for a reference, see for instance Section 5 in [6]. Then, for every $1 \leq i \leq N$, there exists $p_i < \infty$ such that $\mu \in A_{p_i}^{\mathcal{D}_i}(\nu)$, see [5, Theorem 9.33(f)]. In particular $\mu \in A_p^{\mathcal{D}_i}(\nu)$ for $p := \max p_i$, and hence $\|M_\nu^{\mathcal{D}_i}\|_{L^p(\mu) \rightarrow L^p(\mu)} < \infty$ for $1 \leq i \leq N$. It follows that $\|M_\nu\|_{L^p(\mu) \rightarrow L^p(\mu)} < \infty$, which is one possible definition for $\mu \in A_\infty(\nu)$. For much more information, see [5, Section 9.11].) This proves the "continuous" part of Theorem 1.8.

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